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Wigner-Type Theorem on Symmetry Transformations in Type II Factors

Lajos Molnár[1]

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Wigner's theorem on symmetry transformations can be formulated in the following way. If ϕ is a bijective map on the set of all nonzero minimal projections in a type I factor $\mathcal{A}$ which preserves transition probabilities with respect to a faithful normal semifinite trace, then it can be extended to a linear *-automorphism or to a linear *-antiautomorphism of $\mathcal{A}$. In this paper we prove a natural analogue of this statement for type II factors.

Wigner's theorem on symmetry transformations plays fundamental role in quantum mechanics. It has several equivalent formulations. For example, it can be stated in the following form.

Wigner's Theorem. Let H be a complex Hilbert space and denote by $P_1(H)$ the set of all rank-one projections on H. If $\phi: P_1(H) \to P_1(H)$ is a bijective function for which

$$\operatorname{tr} \phi(P)\, \phi(Q) = \operatorname{tr} PQ, \qquad P, Q \in P_1(H) \tag{1}$$

then there exists either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(P) = UPU^*, \qquad P \in P_1(H)$$

In the language of von Neumann algebras one can reformulate Wigner's theorem as follows. Let $\mathcal{A}$ be a type I factor with faithful normal semifinite trace (or, in the terminology of ref. 4, tracial weight) ρ. Denote by $\mathcal{P}_a$ the set of all nonzero minimal projections in $\mathcal{A}$. If $\phi: \mathcal{P}_a \to \mathcal{P}_a$ is a bijective function with the property that

[1] Institute of Mathematics, Lajos Kossuth University, 4010 Debrecen, Hungary; e-mail: molnarl@math.klte.hu

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$$\rho(\phi(P)\phi(Q)) = \rho(PQ), \qquad P, Q \in \mathcal{P}_a$$

then ϕ can be extended to a linear *-automorphism or to a linear *-antiautomorphism of $\mathcal{A}$.

The aim of this paper is to prove a similar statement for type II factors. Since in type II factors, the finite projections play, in some sense, the same role as the minimal projections do in type I factors, it is tempting to formulate the following statement. We note that other Wigner-type results for different structures can be found in our recent papers [6–8].

Theorem. Let $\mathcal{A}$ be a type II factor and let ρ be a faithful normal semifinite trace on $\mathcal{A}$. Denote by $\mathcal{P}_f$ the set of all nonzero finite projections in $\mathcal{A}$. Suppose that $\phi\colon \mathcal{P}_f \to \mathcal{P}_f$ is a bijective function for which

$$\rho(\phi(P)\phi(Q)) = \rho(PQ), \qquad P, Q \in \mathcal{P}_f$$

Then there is either a linear *-automorphism or a linear *-antiautomorphism Φ of $\mathcal{A}$ such that

$$\phi(P) = \Phi(P), \qquad P \in \mathcal{P}_f$$

Proof. If $P \in \mathcal{P}_f$, then by ref. 4, 8.5.2, Proposition, we have $\rho(P) < \infty$. So, it follows from ref. 4, 8.5.1, Proposition, that $\rho(PQ)$, $\rho(\phi(P)\rho(Q))$ are defined. We assert that for any $P, Q \in \mathcal{P}_f$ we have $PQ = 0$ if and only if $\rho(PQ) = 0$. Indeed, if $\rho(PQ) = 0$, then $\rho(QPPQ) = \rho(QPQ) = \rho(PQQ) = \rho(PQ) = 0$. This gives us that $(PQ)^*(PQ) = 0$, which yields $PQ = 0$. Consequently, ϕ preserves the orthogonality between the elements of $\mathcal{P}_f$.

We next extend ϕ to an orthoadditive transformation Φ on the set of all projections in $\mathcal{A}$. This means that $\Phi(P + Q) = \Phi(P) + \Phi(Q)$ holds for any projections $P, Q \in \mathcal{A}$ with $PQ = 0$. Since $\mathcal{A}$ is a factor, every two projections in $\mathcal{A}$ are comparable. As $\mathcal{A}$ is of type II, it contains a nonzero finite projection. We deduce that every nonzero projection in $\mathcal{A}$ has a nonzero finite subprojection. This gives us that every projection in $\mathcal{A}$ is the sum of a system of pairwise orthogonal finite projections. Now, let $P \in \mathcal{A}$ be any projection. If (P_α) is a system of pairwise orthogonal finite projections whose sum is P, then we define

$$\Phi(P) = \sum_\alpha \phi(P_\alpha)$$

This sum is defined since ϕ preserves orthogonality. We show that Φ is well-defined. If (Q_β) has the same properties as (P_α) above and $R \in \mathcal{A}$ is any finite projection, then, by the normality of ρ, we infer

$$\rho(\phi(R)\sum_{\alpha}\phi(P_{\alpha})\phi(R)) = \sum_{\alpha}\rho(\phi(R)\phi(P_{\alpha})\phi(R)) = \sum_{\alpha}\rho(\phi(P_{\alpha})\phi(R))$$
$$= \sum_{\alpha}\rho(P_{\alpha}R) = \sum_{\alpha}\rho(RP_{\alpha}R) = \rho(R\sum_{\alpha}P_{\alpha}R)$$

Similarly, we have

$$\rho(\phi(R)\sum_{\alpha}\phi(Q_{\beta})\phi(R)) = \rho(R\sum_{\beta}Q_{\beta}R)$$

Therefore, we obtain that

$$\rho(\phi(R)\sum_{\alpha}\phi(P_{\alpha})\phi(R)) = \rho(\phi(R)\sum_{\alpha}\phi(Q_{\beta})\phi(R))$$

holds for every finite projection R in $\mathcal{A}$. Since ϕ maps onto $\mathcal{P}_f$, it follows that, with the notation $P' = \Sigma_{\alpha}\,\phi(P_{\alpha})$ and $Q' = \Sigma_{\beta}\phi(Q_{\beta})$, we have

$$\rho(RP'R) = \rho(RQ'R)$$

for every finite projection R in $\mathcal{A}$. We claim that this implies that $P' = Q'$. To verify this, let $R \leq P'$ be a finite subprojection. We obtain that $RQ'R \leq RIR = R = RP'R$. Since R is a finite projection, it follows that $\rho(RP'R)$, $\rho(RQ'R) < \infty$. As $\rho(RP'R) = \rho(RQ'R)$, by the faithfulness of ρ we infer from $\rho(R(P' - Q')R) = \rho(RP'R) - \rho(RQ'R) = 0$ that $R = RP'R = RQ'R$ for every finite subprojection R of P'. We assert that this implies $R \leq Q'$. Indeed, we have $Q'RQ'R = Q'(RQ'R) = Q'R$, showing that $Q'R$ is an idempotent. On the other hand, $\|Q'R\| \leq \|Q'\|\cdot\|R\| = 1$, so $Q'R$ is a contractive idempotent. But this implies that $Q'R$ is a projection and hence we have $Q'R = (Q'R)^* = RQ'$ and we get $Q'R = RQ' = RQ'R = R$. This yields that $R \leq Q'$. Since R is an arbitrary finite subprojection of P', we obtain that $P' \leq Q'$. Interchanging the role of P' and Q', we get the opposite inequality $Q' \leq P'$. Therefore, $P' = Q'$ and hence Φ is well defined.

Clearly, Φ is orthoadditive on the set of all projections. By the solution of the Mackey–Gleason problem [1], Φ can be extended to a bounded linear operator on $\mathcal{A}$. Denote this extension by the same symbol Φ. Since Φ sends projections to projections, it is a standard algebraic argument to show that Φ is a Jordan *-homomorphism (see, for example, the proof of ref. 5, Theorem 2). Since ϕ maps onto $\mathcal{P}_f$, we obtain that the range of Φ contains the set of all projections. Since Φ is a positive linear map on a unital C^*-algebra, it follows that Φ is norm-continuous. By ref. 2, 5.3, Theorem, every closed Jordan ideal in a C^*-algebra is an (associative) ideal. Therefore, Φ induces an injective Jordan *-homomorphism on the quotient C^*-algebra $\mathcal{A}/\ker\Phi$. Now, it follows from ref. 9, Corollary 3.5 that the range of Φ is closed. Since $\mathcal{A}$ is linearly generated by the set of all of its projections in the norm topology, we obtain that Φ is surjective. We show that Φ is injective as well. Let $B \in$

$\mathscr{A}$ be a positive operator in the kernel of Φ. This kernel is an ideal and the values of spectral integrals of bounded Borel functions with respect to the spectral measure corresponding to B belong to $\mathscr{A}$. Multiplying B with an appropriate such spectral integral, we see that the spectral measure E of any Borel subset of the spectrum of B which is in a positive distance from 0 belongs to the kernel of Φ. That is, we have $\Phi(E) = 0$. If $E \neq 0$, then E has a nonzero finite subprojection P. From $\phi(P) = \Phi(P) \leq \Phi(E) = 0$ we have $\phi(P) = 0$, which is a contradiction. So, $E = 0$ and by spectral theorem we conclude that $B = 0$. Suppose now that $\Phi(A) = 0$ for some $A \in \mathscr{A}$. We have $\Phi(A^* A + AA^*) = \Phi(A)^*\Phi(A) + \Phi(A)\Phi(A)^* = 0$. Since $A^*A + AA^*$ is a positive operator belonging to the kernel of Φ, it follows that it is 0, which gives us that $A = 0$. This proves the injectivity of Φ.

It is well known that every factor is a prime algebra, that is, $A\mathscr{A}B = \{0\}$ implies that $A = 0$ or $B = 0$ ($A, B \in \mathscr{A}$). By a classical theorem of Herstein [3], every Jordan homomorphism onto a prime algebra is either a homomorphism or an antihomomorphism. This completes the proof of our theorem. ■

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